

Short Note

A simple, exact solution for the P-SV wave conversion point via prestack migration

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INTRODUCTION

The conversion point (CP) is the lateral position of the reflection point, relative to the source-receiver midpoint, for a laterally homogeneous earth (horizontal reflectors) and the P-SV converted-wave (C-wave) geometry depicted in Figure 1. The source emits a P-wave, which propagates to the horizontal reflector where it reflects as an SV-wave and then propagates to the receiver. Common conversion point (CCP) binning and stacking are tools that use CP modeling for processing C-wave data. Applications include iterative statics and velocity analysis procedures. Acquisition design methods also use CP modeling (see Thomsen, 1999). Calculation of the CP is complicated, unfortunately, and exact solutions exist only for homogeneous and isotropic media. Thomsen (1999) developed asymptotic expansions that approximate the CP in vertically inhomogeneous isotropic and anisotropic media. Yuan and Li (2001) extended these expressions, improving their accuracy for large source-receiver offsets. Exact solutions for the CP exist only in simple media, and these are the subject of this paper. Tessmer and Behle (1988) used Fermat's principle to find a quartic equation for the exact CP in homogeneous, isotropic media. I show that the equation for the C-wave prestack migration impulse response yields a cubic equation for the exact CP in homogeneous, isotropic media. Analytically solving the cubic equation is less cumbersome than analytically solving the quartic equation, and I have obtained the exact solution of the new cubic equation in simple trigonometric form. The Tessmer and Behle (1998) quartic CP equation depends upon the reflector depth, whereas the new cubic CP equation depends instead upon the two-way reflection traveltime of the C-wave. Thus, the new result directly specifies the CP for each sample of a prestack seismic trace (assuming reflectors are horizontal).

PRESTACK C-WAVE MIGRATION AND THE CONVERSION POINT

The C-wave traveltime for the geometry of Figure 1 is given by the double square-root equation

$$T = \sqrt{(x+h)^2 + z^2} \cdot \frac{1}{V_P} + \sqrt{(x-h)^2 + z^2} \cdot \frac{\gamma}{V_P}, \quad (1)$$

where T is two-way C-wave reflection traveltime, z is depth, V_P is P-wave velocity, V_S is S-wave velocity, and $\gamma = V_P/V_S$. The signed source-receiver half-offset is $h \equiv (x_r - x_s)/2$, where x_s and x_r are the source and receiver positions, respectively. The source-receiver midpoint is at $x = 0$. Fermat's principle for horizontal reflectors, $dT/dx = 0$, yields a quartic equation for the CP with parameters h , z , and γ , which is equivalent to the result of Tessmer and Behle (1988).

Alternatively, the double square-root equation (1) may be solved for z using the method of Claerbout (1985, 170). Square equation (1), move the term containing radicals to the left-hand side and all other terms to the right-hand side, and square again. Then, change variables using $y^2 = x^2 + h^2 + z^2$ to get an equation that is quadratic in y^2 . Apply the quadratic formula and simplify to get the prestack C-wave migration impulse response,

$$z^2 = \frac{1}{\beta^2} \cdot [\gamma V_P T - d(x)]^2 - (x-h)^2, \quad (2)$$

where $\beta = 1 - \gamma^2$ and $d(x) = [V_P^2 T^2 - 4h\beta x]^{1/2}$. [See Alfaraj and Larner (1992) for an equivalent, but less compact version of equation (2).] Migration maps the amplitude at time T , from a trace with half-offset h , into the image space along a curve described by the impulse response. Each position on the impulse response represents the location of a possible reflector that is locally tangent to the impulse response. The place on the migration impulse response where the slope is zero represents the location of the CP, or the reflection from a horizontal reflector (see Figure 2). Thus, applying the zero-slope condition, $dz/dx = 0$, to the migration impulse response equation (2) yields the cubic equation for the CP:

$$x^3 + px^2 + qx + r = 0, \quad (3)$$

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where

$$p = \frac{-1}{4h\beta} \cdot [V_p^2 T^2 + 8h^2(\beta - 2)],$$

$$q = \frac{(\beta - 2)}{2\beta^2} \cdot [V_p^2 T^2 + 2h^2(\beta - 2)],$$

and

$$r = \frac{-hV_p^2 T^2}{4\beta}.$$

The cubic equation (3) for x has three real and unequal roots and, after considerable algebra, its analytic solution may be written in the standard trigonometric form (see Burington, 1973, 13),

$$x = \frac{12h^2(\gamma^2 + 1) - C[2 \cos(\phi/3 + 2\pi k/3) + 1]}{12h(\gamma^2 - 1)}, \quad (4)$$

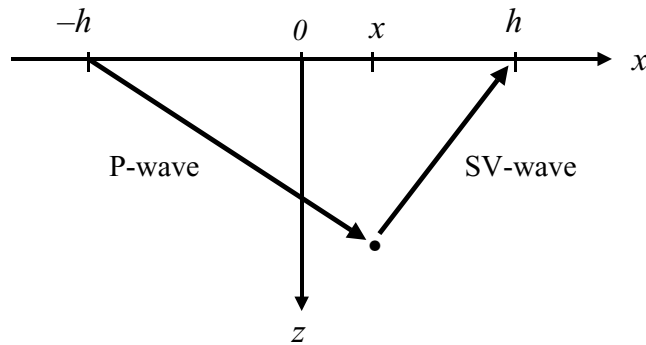


FIG. 1. Source-receiver geometry for P-SV wave conversion. The signed half source-receiver offset is $h = (x_r - x_s)/2$, where x_s and x_r are the source and receiver positions, respectively. The source and receiver are located at lateral coordinates $-h$ and h , respectively; the source-receiver midpoint defines the origin of coordinates; and the P-SV wave conversion point is x .

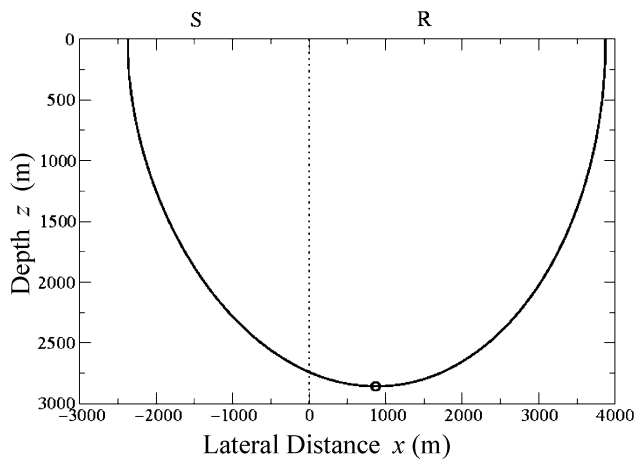


FIG. 2. P-SV wave migration impulse response curve for $h = 1500$ m, $V_p = 2500$ m/s, $\gamma = 3.0$, and $T = 5.0$ s. The circle plotted on the base of the curve, where $dz/dx = 0$, shows the location of the P-SV wave conversion point for a horizontal reflector. The symbols S and R locate the source and receiver, respectively, and the vertical dotted line locates the source-receiver midpoint.

where $C = V_p^2 T^2 + 4h^2(\gamma^2 + 1)$ and

$$\phi = \cos^{-1} \left\{ \frac{C^3 - 864h^4\gamma^2 V_p^2 T^2}{C^3} \right\},$$

and the three roots are given by $k = 0, 1$, and 2 . Only the root for $k = 2$ corresponds to a real-valued solution, z , of the impulse response equation (2), and this is the physical solution representing the exact position of the CP. This result is valid for all $h \neq 0$ and $\gamma > 1$, and is subject to the physical constraint that $V_p T \geq 2|h|$, which says that a fixed offset implies a minimum traveltime for which waves can travel from the source to the receiver. (Alternatively, $|h| \leq V_p T/2$ says there is a

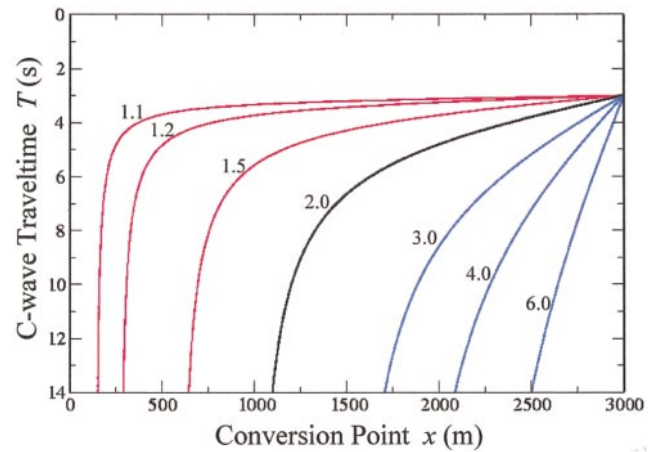


FIG. 3. CP curves plotted against T for fixed h and V_p . Each curve was computed by evaluating the CP equation (4) for $h = 3000$ m, $V_p = 2000$ m/s, and the specific γ value annotated on the plot above that curve. The left three red curves have $\gamma = 1.1, 1.2$, and 1.5 ; the central black curve has $\gamma = 2.0$; and the right three blue curves have $\gamma = 3.0, 4.0$, and 6.0 . Conversion at the source-receiver midpoint occurs at $x = 0$, and conversion at the receiver location occurs at $x = 3000$ m.

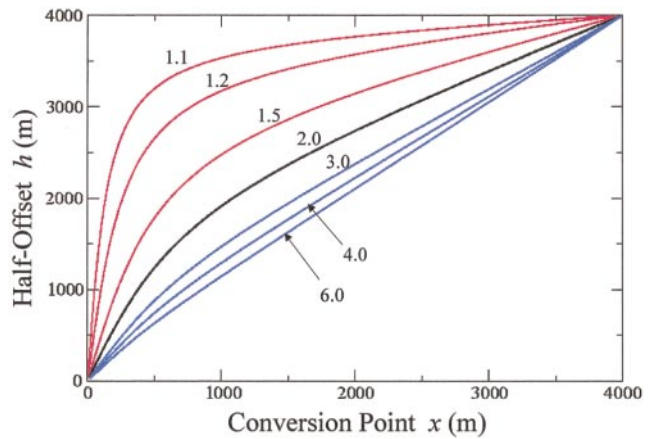


FIG. 4. CP curves plotted against h for fixed T and V_p . Each curve was computed by evaluating the CP equation (4) for $T = 4.0$ s, $V_p = 2000$ m/s, and the specific γ value annotated on the plot above that curve. The top three red curves have $\gamma = 1.1, 1.2$, and 1.5 ; the central black curve has $\gamma = 2.0$; and the bottom three blue curves have $\gamma = 3.0, 4.0$, and 6.0 . Conversion at the source-receiver midpoint occurs at $x = 0$, and conversion at the receiver location occurs on the line, $x = h$.

maximum offset for any fixed traveltime T .) The limiting case, $V_p T = 2|h|$, represents P-wave propagation along the surface, $z = 0$, with P-SV conversion occurring at the receiver location. The inequality constraints also constrain the impulse response equation (2).

LIMITING CASES FOR THE CP SOLUTION

I have examined two limiting cases of the CP solution equation (4). The simplest is its behavior as $h \rightarrow 0$, for which $\phi \rightarrow 0$ and $x \rightarrow h(\gamma^2 + 1)/(\gamma^2 - 1) \rightarrow 0$. This is correct since the conversion point approaches the midpoint as the offset approaches zero. A more difficult case is the behavior of equation (4) as $T \rightarrow \infty$, for which $C \rightarrow \infty$ and $\phi \rightarrow 0$, causing an $\infty \cdot 0$ indeterminacy, which can be resolved through a complicated application of l'Hôpital's rule. The result is the asymptotic limit $x \rightarrow h(\gamma - 1)/(\gamma + 1)$ as $T \rightarrow \infty$. A change of coordinates that shifts the origin from the midpoint to the source location is $X = x + h$. The asymptotic limit in the new coordinates is $X = h(\gamma - 1)/(\gamma + 1) + h = 2h\gamma/(\gamma + 1)$, which equals the asymptotic limit of Tessmer and Behle (1988) and Thomsen (1999).

EXAMPLES

Figure 3 shows a family of CP curves generated by varying γ in CP equation (4) for fixed half-offset $h = 3000$ m, fixed P-wave velocity $V_p = 2000$ m/s, and C-wave traveltime T ranging between 0 and 14 s. The source and receiver were located at -3000 m and $+3000$ m, respectively. The plot shows that for increasing γ at constant T , the CP moves away from the midpoint and toward the receiver location. As $T \rightarrow 2h/V_p = 3.0$ s, the reflector depth implicitly approaches zero, and each curve shows corresponding CP movement to the receiver location.

Figure 4 shows a family of CP curves generated by varying γ in CP equation (4) for fixed C-wave traveltime, $T = 4.0$ s, fixed P-wave velocity $V_p = 2000$ m/s, and half-offset, h varying from 0 to 4000 m. The plot shows that for increasing γ at constant h , the CP also moves toward the receiver location. As $h \rightarrow V_p T/2 = 4000$ m, the reflector depth implicitly approaches zero, and each curve shows corresponding CP movement to the receiver location. Large γ increases CP movement toward the receiver to the extent that for $\gamma = 6.0$, the relationship between the CP and h is almost linear.

VERTICALLY INHOMOGENEOUS MEDIA

A standard approximation for prestack time migration in vertically inhomogeneous media is to replace the velocities in the impulse response equation (2) with their rms (averaged)

values. The CP equation (4) may be extended to vertically inhomogeneous media in the same way. Velocities are usually known, however, in terms of vertical or zero-offset traveltime, T_0 , not in terms of T , and so a mapping from T to T_0 must be known in order to generalize equation (4) to vertically inhomogeneous media. I believe that modified Taylor series of the type described by Tsvankin and Thomsen [1994, their equation (30)] are sufficiently accurate for this purpose.

CONCLUSIONS

Tessmer and Behle (1988) used the depth, z , as an independent variable to find a quartic equation for the exact C-wave CP in isotropic, homogeneous media. I used the independent variable T , the two-way C-wave reflection traveltime, instead of z to find a cubic equation for the exact C-wave CP in isotropic, homogeneous media. Analytically solving a general cubic equation is less cumbersome than analytically solving a general quartic equation, which gives an advantage to using the independent variable T instead of z . I obtained the analytic solution of this cubic equation in the form of a simple trigonometric expression. The new CP expression has limiting values that are equivalent to the Tessmer and Behle (1988) result. Plots of the expression reveal the same behavior as the Tessmer and Behle (1988) result, such as the movement of the CP toward the receiver for increasing $\gamma = V_p/V_s$. Also notable was the almost-linear relationship between the CP and the half-offset, h , for large γ and fixed T .

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