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Linearization of the P-wave eikonal equation for weak vertical transverse isotropy

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ABSTRACT

I develop approximate P-wave and SV-wave eikonal equations for weak vertical transversely isotropic (VTI) media and then use perturbation theory to solve the P-wave eikonal equation. I proceed by expressing the exact VTI eikonal equations for P-waves and SV-waves in terms of the intuitive "Thomsen" anisotropy parameters and then linearizing the results with respect to these parameters. Next, I apply perturbation theory (with a heterogeneous, isotropic "reference" medium) to the P-wave weak VTI eikonal equation. This results in a linear partial differential equation for the traveltime perturbation, which I solve analytically along raypaths in the isotropic reference medium. Traveltimes in the weak VTI medium may then be obtained by simply integrating the reference slowness function plus the (weighted) anisotropy perturbations along raypaths in the isotropic reference medium. This simple solution may be incorporated into standard isotropic ray tracing or traveltime generation methods to produce traveltimes for weak VTI media. I analytically incorporated the method into an isotropic eikonal equation solver and tested it on a vertically heterogeneous weak VTI model, where the strength of the anisotropy was 5–15%. Comparison against results from an exact VTI ray tracing code showed that, for lateral distances up to twice the depth, the weak VTI traveltimes and phase angles were accurate to within 1.3% and 2.2%, respectively. This linear theory yields a physical understanding for how the anisotropy parameters cause traveltimes in VTI media to differ from those in isotropic media. Applications of the theory also include traveltime computation and insights into linear tomographic velocity estimation of reflection seismic data in weak VTI media.

INTRODUCTION

The exact eikonal equations for P- and SV-waves in vertical transverse isotropic (VTI) media are complicated and difficult to use, although Dellinger (1991) showed how to solve them numerically for traveltimes in VTI media. Many numerical solutions to eikonal equations in anisotropic media have subsequently appeared in the literature. The earlier numerical work has been noted in Aldridge and Boneau (1996), and some of the more recent numerical work includes Dellinger and Symes (1997), Ettrich (1998), Kim (1999), and Qian et al. (2001). Alkhalifah (2000a) developed an approximate acoustic P-wave eikonal equation for VTI media by assuming the SV-wave velocity is zero. Alkhalifah (2000b) then linearized this equation with respect to a perturbation of the horizontal P-wave velocity and solved the result with an iterative numerical scheme. Thomsen (1986) wrote "... in most cases of interest to geophysicists the anisotropy is weak (10–20%) ... ", and showed evidence that many sandstones and shales are weakly anisotropic. This has led to several useful weak-anisotropy approximations of the exact equations that describe wave propagation in anisotropic media, particularly VTI media (Thomsen, 1986; Tsvankin and Thomsen, 1994). I apply weak VTI theory in this paper by developing (linearized) eikonal equations for P- and SV-waves in weak VTI media using the intuitive anisotropy parameters of Thomsen (1986). Then, I use perturbation theory to solve the P-wave eikonal equation. My approach parallels the (isotropy) paper of Aldridge (1994) and extends it to VTI media. The present theory provides physical insights for how the anisotropy parameters cause traveltimes in VTI media to differ from traveltimes in isotropic media. Applications of the present theory also include traveltime computation and insights into linear traveltime tomography in weak VTI media.

Linearized solutions of the exact eikonal equation for weak anisotropy have been known at least since Cerveny and Jech (1982) published formulas for general anisotropic media. Their results are in terms of the 21 unknown stiffness coefficients of the general anisotropic elastic wave equation. Specialization

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and simplification from the outset for a specific symmetry, however, such as Aldridge (1994) did for isotropic media, yields a better physical understanding of the problem for that symmetry.

Aldridge (1994) showed that if a heterogeneous, isotropic slowness model is decomposed into a reference slowness model plus a small slowness perturbation, then, to first order in the slowness perturbation, the raypaths in the reference model may be used to compute traveltimes in the total model. A local positive slowness perturbation then causes a global traveltime increase for any reference ray that passes through the perturbed region. This concept underlies linear traveltime tomography in isotropic media and also provides a nice physical interpretation for how a local slowness perturbation causes a global traveltime change.

I proceed by expressing the exact VTI eikonal equations for P-waves and SV-waves in terms of the intuitive anisotropy parameters of Thomsen (1986). The Thomsen parameters allow an easier physical understanding of the equations, and results obtained from them, compared to versions written in terms of the elastic stiffness coefficients. Then, I linearize the eikonal equations, with respect to the Thomsen parameters, obtaining approximate eikonal equations for weak VTI media. Using a heterogeneous, isotropic reference medium, I finally apply perturbation theory to the P-wave weak VTI eikonal equation. This results in a linear partial differential equation for the traveltime perturbation, which I solve analytically along raypaths in the isotropic reference medium. P-wave traveltimes in weak VTI media may then be obtained by simply integrating the reference slowness function plus the (three weighted) anisotropy perturbations along raypaths in the isotropic reference medium. The P-wave traveltime perturbation function depends upon raypath angle and so is more complicated than the corresponding one from Aldridge (1994), but the interpretation proceeds analogously. Local anisotropy perturbations represented by positive values of the Thomson parameters ε and δ cause a global traveltime decrease for rays that pass through the perturbed region. Local positive perturbation of the vertical slowness causes a global traveltime increase for rays that pass through the perturbed region.

I show how the general P-wave traveltime perturbation equation of Cerveny and Jech (1982) may be specialized for weak VTI media, then written in terms of the Thomsen parameters and linearized, producing exactly my traveltime perturbation. Thus, the two weak-anisotropy theories agree. My results also imply that P-wave traveltimes in weak VTI media may be determined by using isotropic ray tracing, to compute rays through the true vertical velocity field, and subsequently timing these rays with the weak VTI phase velocity function of Thomsen (1986).

Although the perturbation theory directly shows how isotropic ray tracing may be used to produce P-wave traveltimes in weak VTI media, it may alternatively be incorporated into eikonal equation solvers to produce weak VTI P-wave traveltimes without tracing rays. Ettrich and Gajewski (1998) did this for arbitrary anisotropy by numerically combining the results of Cerveny and Jech (1982) with the eikonal solver of Vidale (1988). I analytically incorporated the weak VTI P-wave traveltime perturbation theory into the isotropic eikonal equation solver of Schneider et al. (1992). My approach was similar to that of Faria and Stoffa (1994), but it was simpler and only approximate, since I did not use the group velocity function. I tested my method on a vertically heterogeneous weak VTI model, where the strength of the anisotropy was 5–15%. Comparison against results produced by an exact VTI raytracing code showed that, for lateral distances up to twice the depth, the traveltimes and phase angles produced by the weak VTI eikonal solver, were accurate to within 1.3% and 2.2%, respectively.

EIKONAL EQUATIONS FOR WEAK VTI

I begin with the exact P- and SV-wave eikonal equations for VTI media of Daley and Hron (1977):

$$2 = K \pm \sqrt{K^2 - 4L}, \tag{1}$$

where

$$K = (A_{11} + A_{44})p_x^2 + (A_{33} + A_{44})p_z^2,$$
$$L = \left(A_{11}p_x^2 + A_{44}p_z^2\right)\left(A_{44}p_x^2 + A_{33}p_z^2\right) - (A_{13} + A_{44})^2 p_x^2 p_z^2.$$

The A_{ij} in equations (1) represent the four density-normalized elastic stiffness coefficients that describe P- and SV-wave propagation in VTI media, i.e., $A_{ij} \equiv C_{ij}/\rho$, where the C_{ij} are the elastic stiffness coefficients in the Voigt notation (Thomsen, 1986). Quantity τ is traveltime, and $p_x = \partial\tau/\partial x$ and $p_z = \partial\tau/\partial z$ are the horizontal and vertical components of the slowness vector, respectively, which is always normal to the wavefront. The $+$ and $-$ signs in eikonal equations (1) represent the separate eikonal equations for P- and SV-waves, respectively. The slowness vector components in eikonal equations (1) vary only with spatial coordinates x and z. Polar symmetry in VTI media, however, allows these equations to be generalized to three dimensions, so that the slowness vector components vary with x, y, and z, by replacing p_x^2 with $(p_x^2 + p_y^2)$. This substitution may be used throughout this paper to extend its results to three dimensions.

The anisotropy parameters of Thomsen (1986) are

$$\varepsilon = \frac{A_{11} - A_{33}}{2A_{33}}, \quad \delta = \frac{(A_{13} + A_{44})^2 - (A_{33} - A_{44})^2}{2A_{33}(A_{33} - A_{44})},$$
$$\alpha_v^2 = A_{33}, \quad \beta_v^2 = A_{44}, \tag{2}$$

where α_v and β_v are the P- and SV-wave vertical velocities, δ is a complicated combination of stiffness coefficients and, in the weak anisotropy limit, ε is the fractional difference between vertical and horizontal P-wave velocities. I obtain eikonal equations for weak VTI media by inserting the parameters of equations (2) into the eikonal equations (1) and then expanding the square root in equations (1) with a linear Taylor series in ε and δ about the "isotropic" point $(\varepsilon, \delta) = (0, 0)$. The resulting weak VTI eikonal equations for P- and SV-waves are

$$2 = \left[\left(\alpha_v^2 + \beta_v^2\right) \pm \left(\alpha_v^2 - \beta_v^2\right)\right]\left(p_x^2 + p_z^2\right) + \frac{2\varepsilon\alpha_v^2 p_x^2\left[\left(p_x^2 + p_z^2\right) \pm \left(p_x^2 - p_z^2\right)\right]}{p_x^2 + p_z^2} \pm \frac{4\delta\alpha_v^2 p_x^2 p_z^2}{p_x^2 + p_z^2}. \tag{3}$$

The weak VTI eikonal equation, for P-waves (+ sign) is

$$\frac{1}{\alpha_{\rm v}^2} = \left(p_x^2 + p_z^2\right) + \frac{2\left(\varepsilon p_x^2 + \delta p_z^2\right) p_x^2}{p_x^2 + p_z^2}, \quad (4)$$

and for SV-waves (− sign) is

$$\frac{1}{\beta_{\rm v}^2} = \left(p_x^2 + p_z^2\right) + \frac{2(\varepsilon - \delta)\left(\alpha_{\rm v}^2/\beta_{\rm v}^2\right) p_x^2 p_z^2}{p_x^2 + p_z^2}. \quad (5)$$

The second term on the right side of weak VTI eikonal equations (4) and (5) represents the anisotropy term that makes these equations different from the (familiar) eikonal equations for isotropic media. Multiplying these equations through by $(p_x^2 + p_z^2)$ shows that they are first-order nonlinear partial differential equations for the traveltime τ, with up to powers of four in the derivatives p_x and p_z [see equation (8)]. The isotropic eikonal equations are less nonlinear, containing only powers of two in p_x and p_z.

The P-wave eikonal equation (4) is independent of β_v, indicating that P-waves in VTI media depend only weakly on the SV-wave velocity. The P-wave equation also shows that the parameters ε and δ are most influential in different angular ranges, with greatest ε influence at large angles (where $p_x \gg p_z$) and greatest δ influence at intermediate angles (where $p_x < p_z$ and $p_x > 0$). Since the P-wave equation reduces to the form of the isotropic eikonal equation for vertical propagation (where $p_x = 0$), neither ε nor δ influence traveltimes for very small angles. The anisotropy term of the SV-wave eikonal equation (5) contains the amplification factor $(\alpha_{\rm v}/\beta_{\rm v})^2$, not found in the P-wave eikonal equation, indicating that SV-wave anisotropy is more pronounced than P-wave anisotropy for small, fixed values of ε and δ. The factor $(\alpha_{\rm v}/\beta_{\rm v})^2\ (\varepsilon - \delta)$ in the anisotropy term of equation (5) is the anisotropy parameter σ that Tsvankin and Thomsen (1994) refer to as "the most influential parameter in the SV-wave velocity and moveout equations." Eikonal equation (5) shows that for SV-waves, parameters ε and δ are most influential at intermediate angles, and the degree to which ε and δ influence traveltimes depends upon their differerence. The SV-wave equation reduces to the form of the isotropic eikonal equation for vertical and horizontal propagation (where the product $p_x p_z = 0$), and when $\varepsilon = \delta$, which is the elliptical anisotropy case. These observations are all consistent with similar findings of Tsvankin and Thomsen (1994).

PERTURBATION THEORY FOR P-WAVES

I linearize the P-wave eikonal equation for weak VTI using perturbation theory analogous to Aldridge (1994) by defining an inhomogeneous and isotropic reference medium, where ε and δ are zero, traveltime is τ_0, slowness is $s_0 = 1/\alpha_0$, and the slowness vector components are $p_{0x} = \partial\tau_0/\partial x$ and $p_{0z} = \partial\tau_0/\partial z$. With these quantities, the P-wave eikonal equation (4) becomes the isotropic eikonal equation for the reference medium,

$$p_{0x}^2 + p_{0z}^2 = \frac{1}{\alpha_0^2} \equiv s_0^2. \quad (6)$$

The weak VTI medium differs slightly from the isotropic reference medium through the three small, spatially variable perturbations, ε, δ, and $s_1 \equiv s_v - s_0$, where $s_v = 1/\alpha_{\rm v}$ is the vertical slowness function for the weak VTI medium and s_1 is the vertical slowness perturbation. I define the traveltime perturbation as τ_1, and the perturbations of the slowness vector components as $p_{1x} = \partial\tau_1/\partial x$ and $p_{1z} = \partial\tau_1/\partial z$. The traveltime and slowness vector perturbations τ_1, p_{1x}, and p_{1z} must also be small because the medium perturbations ε, δ, and s_1 were defined to be small. Thus, quantities for the weak VTI medium are described by the decomposition

$$\begin{aligned} \tau &= \tau_0 + \tau_1, \quad s_v = s_0 + s_1, \\ p_x &= p_{0x} + p_{1x}, \quad p_z = p_{0z} + p_{1z}. \end{aligned} \quad (7)$$

P-wave eikonal equation (4) may be rewritten as

$$\left(p_x^2 + p_z^2\right)^2 + 2p_x^2\left(\varepsilon p_x^2 + \delta p_z^2\right) - s_v^2\left(p_x^2 + p_z^2\right) = 0. \quad (8)$$

Linearization then proceeds in two steps. First, insert the total slowness vector components of equation (7) into P-wave eikonal equation (8) and apply the reference-medium eikonal equation (6). Next, discard all terms that contain products of the five small quantities p_{1x}, p_{1z}, ε, δ, and s_1, i.e., discard terms containing $(p_{1x})^2$, $p_{1x}p_{1z}$, εp_{1x}, $s_1 p_{1z}$, etc. The result is the linear equation for the P-wave traveltime perturbation τ_1 (assuming an isotropic reference medium):

$$p_{0x}p_{1x} + p_{0z}p_{1z} = s_0 s_1 - \frac{1}{s_0^2} p_{0x}^2\left(\varepsilon p_{0x}^2 + \delta p_{0z}^2\right). \quad (9)$$

This equation may be solved analytically by recognizing that the left side is the dot product of the gradient of τ_0 with the gradient of τ_1. A unit vector normal to the wavefronts in the isotropic reference medium is $(1/s_0)\nabla\tau_0 \equiv \boldsymbol{l_0}$, which is everywhere parallel to some raypath Γ in the reference medium. The left side is, therefore, a directional derivative in the direction of the rays in the isotropic reference medium. Thus, equation (9) becomes

$$\begin{aligned} (1/s_0)\nabla\tau_0 \cdot \nabla\tau_1 = \boldsymbol{l_0} \cdot \nabla\tau_1 &= \frac{d\tau_1}{d\ell_0} \\ &= s_1 - \frac{1}{s_0^3} p_{0x}^2\left(\varepsilon p_{0x}^2 + \delta p_{0z}^2\right), \end{aligned} \quad (10)$$

which may be integrated along the isotropic reference ray Γ to obtain the solution

$$\tau_1 = \int_\Gamma d\ell_0 s_1 - \int_\Gamma d\ell_0 \frac{1}{s_0^3} p_{0x}^2\left(\varepsilon p_{0x}^2 + \delta p_{0z}^2\right). \quad (11)$$

The integration curve Γ in traveltime perturbation equation (11) is normal to the wavefronts in the isotropic reference medium, and so it is a raypath in the isotropic medium but not in the associated weak VTI medium. Polar symmetry in VTI media allows equation (11) to be generalized to three dimension by replacing p_{0x}^2 with $(p_{0x}^2 + p_{0y}^2)$ (see previous discussion).

Traveltime perturbation equation (11) may be expressed in terms of the phase angle θ that the isotropic ray Γ makes with respect to the z-axis by using $p_{0x} = s_0 \cdot \sin\theta$ and $p_{0z} = s_0 \cdot \cos\theta$. The result is

$$\tau_1 = \int_\Gamma d\ell_0 s_1 - \int_\Gamma d\ell_0 s_0 \sin^2\theta(\varepsilon\sin^2\theta + \delta\cos^2\theta). \quad (12)$$

Equation (12) shows that ε exerts its greatest influence on τ_1 at large angles, and δ exerts its greatest influence on τ_1 at intermediate angles. Figure 1 illustrates these angle dependencies. It shows that ε exerts little influence on τ_1 at small angles, which is primarily controlled by δ and s_1. In turn, δ exerts little influence on τ_1 for near-vertical propagation (very small angles), where s_1 dominates τ_1, and equation (12) reduces, in form, to that of Aldridge (1994). A local positive vertical slowness perturbation causes a positive contribution to the traveltime perturbation τ_1, increasing the global traveltime for rays that pass through the perturbed region. Local anisotropy perturbations represented by positive values of ε and δ cause angle-dependent negative contributions to the traveltime perturbation τ_1, reducing the global traveltime for rays that pass through the perturbed region.

Connection to Thomsen (1986)

An expression for the total traveltime τ in an inhomogeneous medium where $s_v = s_0$ (and thus $s_1 = 0$) may be obtained by adding the second integral in traveltime equation (12) to the integral for the traveltime τ_0 in the isotropic reference medium. The result is

$$\begin{aligned}\tau &= \int_\Gamma dl_0 s_v - \int_\Gamma dl_0 s_v \sin^2\theta(\varepsilon \sin^2\theta + \delta\cos^2\theta) \\ &= \int_\Gamma dl_0 s_v [1 - \delta \sin^2\theta\cos^2\theta - \varepsilon\sin^4\theta] \equiv \int_\Gamma dl_0/v(\theta),\end{aligned} \tag{13}$$

where

$$\begin{aligned}v(\theta) &= \frac{1}{s_v[1 - \delta\sin^2\theta\cos^2\theta - \varepsilon\sin^4\theta]} \\ &\approx \alpha_v[1 + \delta\sin^2\theta\cos^2\theta + \varepsilon\sin^4\theta].\end{aligned} \tag{14}$$

I use a linear Taylor series expansion in ε and δ to make the approximation in equation (14). Velocity equation (14) equals the weak VTI phase velocity for P-waves of Thomsen [1986, equation (16a)], but expressed in terms of the phase angle θ in the isotropic reference medium. The perturbation theory predicts that raypaths traced in an isotropic reference medium, where the slowness is s_v, may then be "timed" using the weak VTI phase velocity function, to obtain traveltimes in the weak VTI medium. Even in a homogeneous weak VTI medium, where raypaths are straight, the perturbation theory traveltime of equation (13) is only approximate, since the group velocity would correctly time the rays. [Ettrich and Gajewski (1998) also shows that anisotropic perturbation theory is not exact for simple cases, where the raypaths in the reference and perturbed medium are the same.]

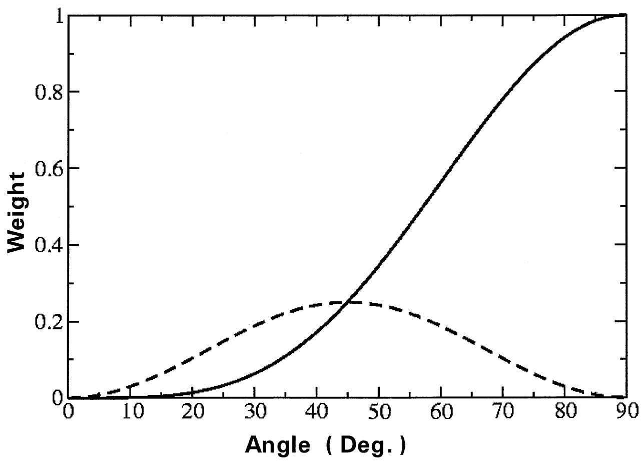

FIG. 1. Angle-dependent weighting functions of $\sin^4\theta$ for the ε term (solid curve) and $\sin^2\theta \cdot \cos^2\theta$ for the δ term (dashed curve) in the traveltime perturbation equation (12). The weighting functions show that ε contributions to the traveltime perturbation are weighted to strongly emphasize large angles, whereas δ contributions are weighted to emphasize intermediate angles.

Connection to Cerveny and Jech (1982)

Cerveny and Jech [1982, equation (49)] wrote an expression for the P-wave traveltime perturbation in a general weakly anisotropic medium, where the reference medium is isotropic, which can be specialized to VTI media by setting $A_{15} = A_{35} = 0$. The result is linear in the density-normalized stiffness perturbations and, after changing the integration variable from traveltime to arc length and using $A_{55} = A_{44}$ (valid in VTI media), I write it in the form

$$\tau_1 = -\frac{1}{2}\int_\Gamma d\ell_0 \frac{1}{s_0}\left[A^1_{11}p_1^4 + A^1_{33}p_3^4 + 2\left(A^1_{13} + 2A^1_{44}\right)p_1^2 p_3^2\right]. \tag{15}$$

The four stiffness perturbations in traveltime equation (15) are differences between the stiffness coefficients in the VTI medium and the isotropic reference medium, thus $A^1_{IJ} = A_{IJ} - A^0_{IJ}$. [The A_{IJ} are the actual VTI medium parameters, as in equation (1), and the A^0_{IJ} are the parameters that describe the isotropic reference medium.] Thomsen's parameters of equation (2) may be used to write the A^1_{IJ} in terms of ε, δ, s_1, s_0, and u_0, where u_0 is the SV-wave slowness function for both the reference medium and the VTI medium. (SV-wave slowness is not perturbed, thus $A^1_{44} = 0$.) These A^1_{IJ} functions may then be linearized (using Taylor series expansions) in the variables ε, δ, and s_1 and inserted into equation (15) to yield exactly the traveltime perturbation equation (11). The perturbation equation (11) naturally depends upon only three medium parameters (ε, δ, and s_1) because the weak VTI eikonal equation (4) does not depend upon the SV-wave vertical velocity. The relatively complicated equation (15) depends upon four medium parameters, but can be made to depend upon three parameters by explicitly setting the SV-wave perturbation A^1_{44} to zero.

IMPLICATIONS FOR TOMOGRAPHY

The weak VTI traveltime perturbation equation (12) and Figure 1 show how the parameters ε and δ influence P-wave traveltimes in different angular ranges, with greatest ε influence at large angles and greatest δ influence at intermediate angles. Neither parameter plays a role for vertical propagation, which is entirely controlled by the P-wave vertical slowness perturbation s_1. The traveltime perturbation equation (12) could be used for the tomographic inversion of surface seismic reflection P-wave data to estimate s_1, ε, and δ, but there are some pitfalls for such a process.

Raypath angles corresponding to reflected waves, in a horizontally stratified earth, naturally decrease as depth increases for fixed acquisition geometry. When raypath angles are small, the ε term in traveltime perturbation equation (12) becomes small and insignificant for a wide range of ε values. The tomographic resolution of ε will thus decrease with depth in such a (typical) situation; however, this depth resolution should improve with longer offset seismic data.

Tsvankin and Thomsen (1995) wrote that there is a "trade-off between vertical velocity and the parameters of anisotropy on gathers with limited angle coverage", and concluded that α_v, ε, and δ cannot be uniquely determined solely from surface seismic P-wave data. (Recall that $s_v = 1/\alpha_v$ and $s_1 = s_v - s_0$.) Different values for s_1, ε, and δ can produce the same traveltime perturbation τ_1 from equation (12), particularly in regions of the model where the anisotropy parameters are not well resolved (e.g., large depths or small angles). Tsvankin and Thomsen (1995) indicated that the true vertical velocity and the anisotropy parameters may be determined in the vicinity of wells by combining well data and normal moveout velocities. The inversion could then be constrained in the vicinity of the wells to reduce nonuniqueness. The term containing δ in traveltime perturbation equation (12) is more influential for small raypath angles than the term involving ε (see Figure 1), and the term involving s_1 is the most influential term for very small angles. [Similar observations were made by Thomsen (1986).] This suggests that the parameters s_1 and δ are largely responsible for the trade-off discussed by Tsvankin and Thomsen (1995). Tomographic inversion of prestack reflection seismic data particularly exploits the moveout of seismic traces in common-image-point gathers (Docherty et al., 2000). The weak VTI normal moveout velocity function, for P-waves in a homogeneous layer, of Thomsen (1986) is $V_{NMO} = \alpha_v(1+\delta)$. For this simple case, the moveout depends upon the two parameters α_v and δ in a similar (angle-independent) way, also suggesting a trade-off of α_v (or s_1) and δ for a tomographic process that senses the moveout of seismic data.

TRAVELTIME COMPUTATION

The perturbation theory shows how P-wave traveltimes in a weak VTI medium may be approximated by a method based upon isotropic ray tracing. A raypath and its traveltime τ_0 are first determined in the inhomogeneous, isotropic reference medium. Then a traveltime perturbation τ_1 is obtained by integrating the three small, spatially variable parameters ε, δ, and s_1 along the reference medium raypath. The traveltime in the weak VTI medium is then approximated by $\tau = \tau_0 + \tau_1$. This process may be repeated for a set of rays that cover the reference medium, thereby producing approximate traveltimes throughout the VTI medium. The traveltime perturbation equations may also be incorporated into existing isotropic eikonal equation solvers to obtain weak VTI media traveltimes without ray tracing. Ettrich and Gajewski (1998) did this by numerically combining the perturbation theory of Cerveny and Jech (1982) with the eikonal solver of Vidale (1988). Their method estimates raypath segments from the traveltimes within each grid cell and uses these raypath segments to compute traveltime perturbations for the weakly anisotropic medium. They used both isotropic and elliptically anisotropic reference media and indicated that elliptically anisotropic reference media allow the method to handle stronger anisotropy.

Alternatively, I analytically incorporated the P-wave traveltime perturbation theory for weak VTI media into the isotropic eikonal equation solver of Schneider et al. (1992). The details are in the Appendix. My approach was similar to that of Faria and Stoffa (1994), but it was simpler and only approximate, since I did not use the group velocity function.

I compared the output of this new weak VTI eikonal solver with the output of an exact VTI ray tracing code using the horizontally layered earth model of Table 1. Figures 2 and 3 are

Table 1. Parameters for the five-layer earth model used to test the weak VTI eikonal solver. The depth to the base of each layer is Z, and the three Thomsen parameters for each layer are α_v, ε, and δ.

Z (m)	α_v (m/s)	ε	δ
304.8	1524.0	0.0	0.0
762.0	1828.8	0.10	0.05
914.4	2133.6	0.15	−0.10
1280.2	1981.2	0.05	0.10
1524.0	2286.0	0.12	0.05

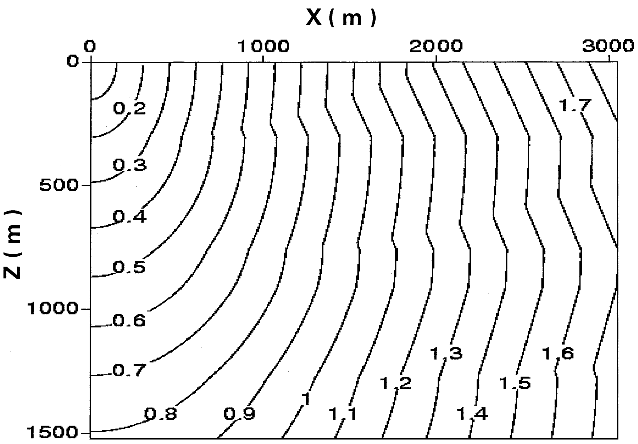

FIG. 2. Traveltime function generated by the weak VTI eikonal solver using the isotropic reference medium, which consists of the model of Table 1 but with $\varepsilon = \delta = 0$. The source point is at the origin, and the contours represent first-arrival wavefronts. The contour interval is 0.1 s.

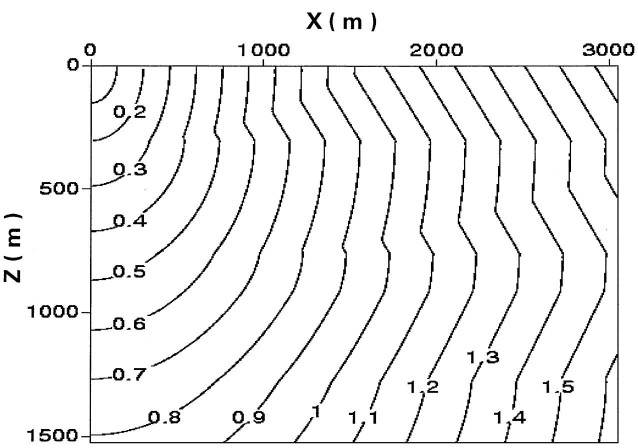

FIG. 3. Traveltime function generated by the weak VTI eikonal solver using the weak VTI model of Table 1. The source point is at the origin, and the contours represent first-arrival wavefronts. The contour interval is 0.1 s.

contour plots of the traveltimes obtained with the weak VTI eikonal solver for the isotropic reference model ($\varepsilon = \delta = 0$) and the anisotropic model, respectively. The contours in these plots represent first arrival wavefronts. The traveltimes for the two models are similar for small angles, as equation (12) predicts, but they exhibit large differences of up to 0.1 s for large angles. Figure 4 shows traveltimes at depth 1524 m from Figures 2 and 3 and traveltimes obtained by an exact VTI ray-tracing code. The curves show traveltimes for lateral distances up to twice the depth, and the maximum error in the weak VTI traveltimes is just 1.3%. Figure 5 is analogous to Figure 4, except it shows phase angles instead of traveltimes, where the phase angle is the angle that the gradient of the traveltime makes with the vertical. The maximum error in the weak VTI phase angles is about 2.2%. Figure 5 shows that the weak VTI phase angles, estimated numerically from the traveltimes of Figure 3, agree well with the phase angles used by the exact VTI ray-tracing code.

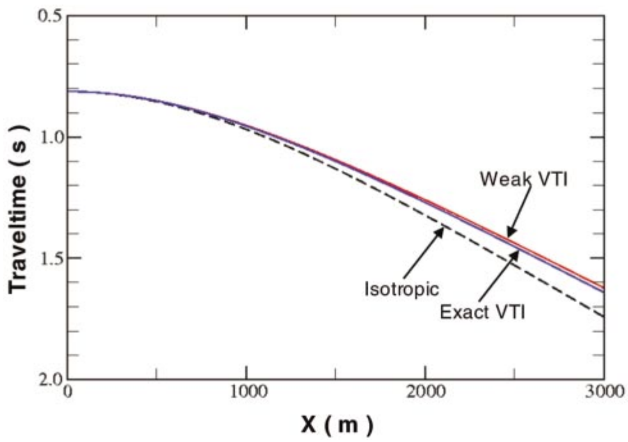

FIG. 4. Traveltime curves taken at depth 1524 m from Figure 2 for the isotropic reference medium (dashed black) and from Figure 3 for the weak VTI medium (solid red). Exact VTI ray tracing to the depth of 1524 m in the model of Table 1 generated the solid blue traveltime curve.

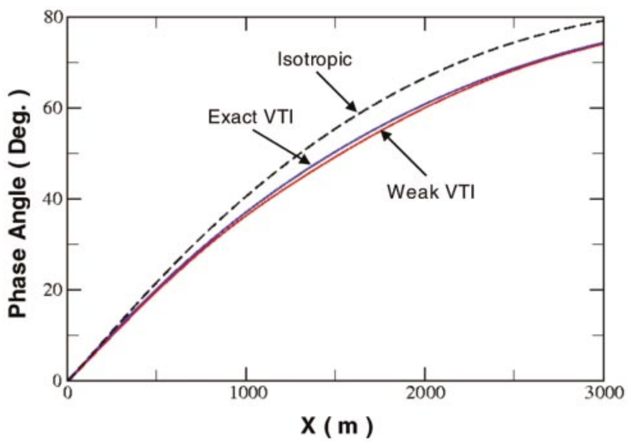

FIG. 5. Phase angle curves estimated numerically at depth 1524 m from Figure 2 for the isotropic reference medium (dashed black) and from Figure 3 for the weak VTI medium (solid red). Exact VTI ray tracing to the depth of 1524 m in the model of Table 1 used the phase angles shown by the solid blue curve.

CONCLUSIONS

I present two main results in this paper. First are the P-wave and SV-wave eikonal equations (4) and (5) for weak VTI media, which I obtain by linearizing the exact VTI eikonal equations with respect to the intuitive Thomsen parameters. These results are easier to interpret physically than the exact eikonal equations because they only differ from isotropic eikonal equations through one term that is linear in ε and δ. Second is the traveltime perturbation result of equations (11) and (12), which I obtain using the method of Aldridge (1994). The traveltime perturbation function for VTI media is more complicated than the one in Aldridge (1994), being a function of raypath angle, but its physical interpretation is analogous. Local anisotropy perturbations, represented by positive values of ε and δ, cause a global traveltime decrease, and a local positive vertical slowness perturbation causes a global traveltime increase, for any reference ray that passes through the perturbed region. Thus, positive ε and δ (perturbations) reduce traveltimes, and a positive vertical slowness perturbation increases traveltimes, relative to the traveltimes in the reference medium.

I show how the general P-wave traveltime perturbation equation of Cerveny and Jech (1982) may be written in terms of the Thomsen parameters and then linearized, producing exactly my traveltime perturbation equation (11). Thus, the two weak-anisotropy theories agree. The traveltime perturbation theory also shows that P-wave traveltimes in weak VTI media may be determined by using isotropic ray tracing to compute rays through the true vertical velocity field and, subsequently, timing these rays with the weak VTI phase velocity function of Thomsen (1986).

One application of this theory is a simple method, based upon ray tracing in an isotropic reference medium, for modeling P-wave traveltimes in weak VTI media. The theory may alternatively be incorporated into isotropic eikonal equation solvers to model accurate P-wave traveltimes and phase angles in weak VTI media without ray tracing. I did this analytically by actually modifying the governing equations of an existing isotropic eikonal equation solver. Traveltimes and phase angles from the resulting weak VTI eikonal solver code closely matched those generated by exact VTI ray tracing for a weak VTI earth model.

My use of a heterogeneous, isotropic reference model for the perturbation theory is a simple, convenient approach that solves the weak VTI anisotropy problem for P-waves. An elliptically anisotropic reference model may lead to better accuracy, however, as indicated by Ettrich and Gajewski (1998), particularly for cases where the anisotropy is not weak.

The weak VTI traveltime perturbation equation (12) shows that the parameters ε and δ influence P-wave traveltimes in different angular ranges, with greatest ε influence at large angles and greatest δ influence at intermediate angles. Neither parameter contributes to vertical propagation, which is entirely controlled by the P-wave vertical slowness perturbation. The traveltime perturbation equation (12) could be used for the tomographic inversion of surface seismic reflection P-wave data to estimate s_1, ε, and δ, but there are some pitfalls for such a process. These include (likely) reduced resolution of ε as depth increases and a trade-off between the parameters δ and s_1. Different values for s_1, ε, and δ can thus produce the same traveltimes, particularly in regions of the model where the anisotropy parameters are not well resolved (e.g., large depths or small angles). Constraining an inversion with well data has been suggested as a way of reducing the nonuniqueness (Tsvankin and Thomsen, 1995).

Perturbation theory for P-waves in VTI media has several useful applications that have been the focus of this paper. Perturbation theory may also be applied to the SV-wave eikonal equation (5) to obtain analogous results for SV-waves in weak VTI media. SV-waves in VTI media generally exhibit cusps, however, which would not be modeled properly by the SV-wave perturbation theory because of its isotropic reference medium. But since SV-wave cusp formation is insignificant in weak VTI media (Tsvankin and Thomsen, 1995; Schoenberg and de Hoop, 2000), the perturbation theory should also be useful for SV-waves in the context of weak VTI.

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APPENDIX

FERMAT'S PRINCIPLE TRAVELTIME CALCULATIONS FOR WEAK VTI

Schneider et al. (1992) describe an exhaustive search "brute force" procedure to generate a grid of traveltimes in isotropic media. At each step of the procedure, the method propagates the traveltime field through a homogeneous grid cell by locally satisfying Fermat's principle. In this Appendix, I analytically incorporate the P-wave traveltime perturbation theory into the application of Fermat's principle used by that method, extending it to weak VTI media. I proceed by first writing an expression for the traveltime of a raypath segment that traverses the homogeneous grid cell of Figure A-1. [Also see Figures 2 and 3a of Schneider et al. (1992).] This may be done for weak VTI by specializing the traveltime perturbation equation (12) to a homogeneous medium with no slowness perturbation, where $s_v = s_0$ and thus $s_1 = 0$, and simply adding the resulting perturbation,

$$-r s_v \sin^2\theta(\varepsilon \sin^2\theta + \delta \cos^2\theta),$$

to the expression for traveltime of Schneider et al. [1992, equation (2)], resulting in

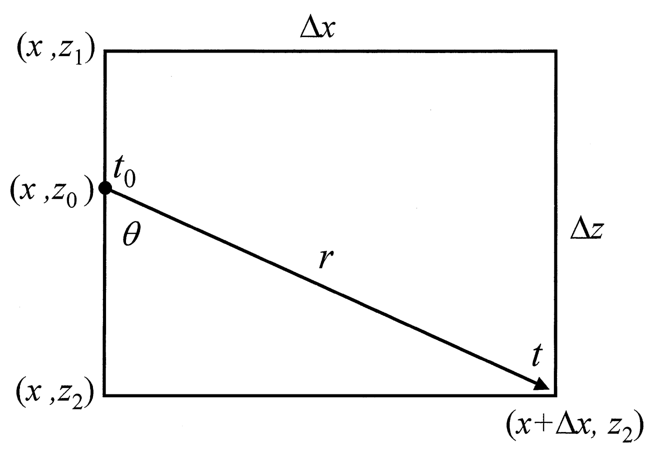

FIG. A-1. Horizontal cell configuration for a homogeneous grid cell used in the traveltime calculations. A known traveltime function $t_0(z_0)$, specified on the left cell edge between the depths z_1 and z_2, will be used to compute a traveltime t at the lower right cell corner, which satisfies Fermat's principle.

$$t = t_0 + r s_v[1 - \sin^2\theta(\varepsilon \sin^2\theta + \delta\cos^2\theta)]. \quad \text{(A-1)}$$

Traveltime t_0 is the known traveltime function on the vertical cell edge, evaluated at depth z_0, which I specify by the nonlinear interpolation formula of Schneider et al. [1992, equation (7)]. The quantity r is the distance traveled by the raypath through the homogeneous cell, and I reference the angle θ to the vertical, so that $\cos\theta = (z_2 - z_0)/r$ and $\sin\theta = \Delta x/r$. Fermat's principle applied to equation (A-1) for the "vertical cell configuration" of Figure A-1 then yields

$$\frac{\partial t}{\partial z_0} = 0 = \frac{\partial t_0}{\partial z_0} - \frac{(z_2 - z_0)s_v}{r^5} \times [r^4 + 3\varepsilon\Delta x^4 + \delta\Delta x^2(r^2 - 3\Delta x^2)]. \quad \text{(A-2)}$$

Fermat's principle for the "horizontal cell configuration" of Figure A-2 [and Figure 3b in Schneider et al. (1992)] must be calculated separately for the VTI case so that θ remains referenced to the vertical. In this case, $\cos\theta = \Delta z/r$ and $\sin\theta = (x_2 - x_0)/r$, and traveltime t_0 analogously becomes the known traveltime function on the horizontal cell edge, evaluated at lateral position x_0. Fermat's principle applied to equation (A-1) for the horizontal cell configuration then yields,

$$\frac{\partial t}{\partial x_0} = 0 = \frac{\partial t_0}{\partial x_0} - \frac{(x_2 - x_0)s_v}{r^5}\big[r^4 - \varepsilon(x_2 - x_0)^2(r^2 + 3\Delta z^2) + \delta\Delta z^2(r^2 - 3\Delta z^2)\big]. \quad \text{(A-3)}$$

Fermat's principle equations (A-2) and (A-3) for weak VTI replace the Fermat's principle equation of Schneider et al. [1992, equation (8)] and generalize that method to weak VTI media. Separate Fermat's principle calculations for the vertical and horizontal cell configurations were not necessary in Schneider et al. (1992) because the angle θ did not appear in the isotropic problem. In that case, Fermat's principle for one cell configuration could be transformed to the other by simply interchanging x and z variables.

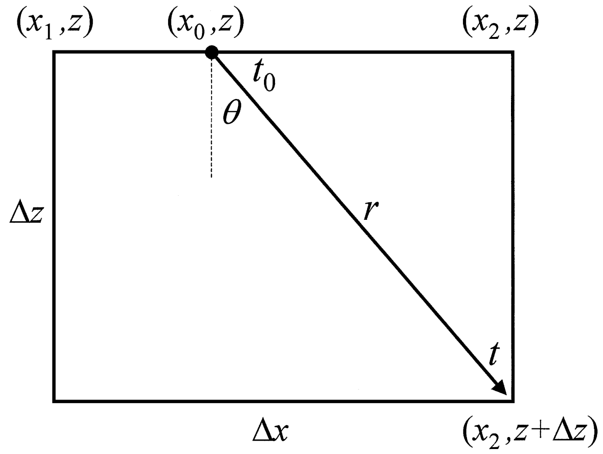

FIG. A-2. Vertical cell configuration for a homogeneous grid cell used in the traveltime calculations. A known traveltime function $t_0(x_0)$, specified on the upper cell edge between the positions x_1 and x_2, will be used to compute a traveltime t at the lower right cell corner, which satisfies Fermat's principle.